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EXPLICIT SOLUTION AND INVARIANCE OF THE SINGULARITIES AT AN INTERFACE CRACK IN ANISOTROPIC COMPOSITES

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Abstract—Assuming that stress distribution near the tip of an interface crack in an anisotropic composite is proportional to r^δ, application of the interface and boundary conditions yields $\|\mathbf{K}(\delta)\| = 0$, where **K** is a 12×12 complex matrix. The surfaces of the crack can be free–free, fixed–fixed or free–fixed. For the cases of free–free and fixed–fixed cracks, explicit solutions for all δ's are obtained. For the case of free–fixed crack, the determinant of **K** is reduced to a 3×3 determinant which yields a sextic equation. Explicit solutions are obtained only for isotropic composites. The special cases of a homogeneous anisotropic material with a semi-infinite crack and the half-plane problems are also considered. Explicit solutions for δ's are obtained for all three boundary conditions. Finally, it is shown that δ is invariant with respect to the orientation of the plane boundary (in the case of half-plane problems), the semi-infinite crack (in the case of a crack in a homogeneous material) and the crack and interface (in the case of a composite with an interface crack) relative to the materials. This is a somewhat surprising result not expected of anisotropic materials.

1. INTRODUCTION

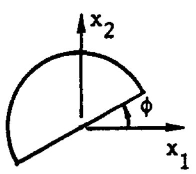
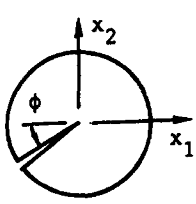
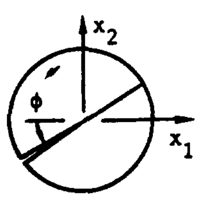
The problem of finding the stress singularities at the apex of an isotropic elastic wedge was first considered by Knein[1] and Williams[2] in which they assume that the stresses near the apex are proportional to r^δ, where r is the radial distance from the apex, and δ is a constant. It is shown that $\delta < 0$ when the wedge angle is larger than π and that the stresses are singular at the apex. The technique is applied to a crack along[3, 4] and normal[5, 6] to the interface and to other geometries of isotropic composites[7–12]. A systematic derivation of the equation for finding the singularity δ was given by Dempsey and Sinclair[12] who also provided a large number of reference to other workers on the problem.

Investigation of the associated problems for anisotropic materials started in [13, 14] and has become active only in the last decade. (See [15–25], for example.) For anisotropic materials, the out-of-plane displacement is in general nonzero even if one assumed a two-dimensional deformation. Therefore, the problem of finding the value of δ for a composite with an interface crack reduces to finding the roots of the determinant of **K** in which **K** is a 12×12 complex matrix. In the case of a crack normal to the interface[20, 21, 25], **K** is an 18×18 complex matrix. The roots δ can be real or complex and in general one has to locate them numerically in the complex plane. For the stress singularities, one is interested in the region where the real part of δ is in the range $-1 < \mathrm{Re}(\delta) < 0$. In many applications, however, δ's with a positive real part are also needed[23, 26, 27]. This means that one has to search numerically for the δ's in the region where $\mathrm{Re}(\delta) > -1$. For the special geometry of composite wedges considered here, explicit analytical solutions can be obtained by using Stroh's formulation for analyzing anisotropic elastic materials.

Stroh's formulation[28, 29], which has its origin in [30], provides an elegant and powerful method of treating a certain class of two-dimensional anisotropic elasticity problems such as dislocations, line forces and steady-state wave propagation. Unlike the two-dimensional anisotropic solutions developed by Green and Zerna[31], which are restricted to plane strain deformations, the Stroh formalism applies to a wide variety of two-dimensional problems in which all three displacement components are nonzero. Also, unlike the widely used Lekhnitskii's approach[32], which breaks down for orthotropic materials[33] and requires a special treatment[23], the Stroh formalism has no limitations except possibly for the degenerate materials in which the eigenvalues of the elasticity constants have a repeated root such as in isotropic materials. The problem with degenerate materials, for which other formalisms also have, can be treated separately[34, 35]. However, the Stroh formalism has since been perfected by Barnett and Lothe[36, 37]. They show that many anisotropic elasticity solutions can be expressed in terms of one or more of the three real matrices, **H**, **L** and **S**, which are, in the notations of [38], $-\mathbf{Q}$, $4\pi\mathbf{B}$ and **S**, respectively. The three real matrices can be determined directly by using the integral formalism of Barnett and Lothe[36] without the need of computing the eigenvalues and eigenvectors of elasticity constants, and thus the problem of repeated eigenvalues disappears. An excellent review on and further developments of the Stroh formalism as well as additional references can be found in [39]. Unfortunately, Stroh's work as well as that of Barnett and Lothe have attracted little attention from researchers in anisotropic composites. Some work in anisotropic composites using the Stroh formalism can be found in [15, 33–35]. In this paper we use the Stroh formalism to obtain explicit solutions to certain stress singularity problems.

We consider a special geometry of anisotropic composite wedges in which the two individual wedges have the wedge angle π. In the polar coordinate system (r, θ), they are glued together along $\theta = \phi$, while $\theta = \phi \pm \pi$ are the crack surfaces which can be traction free or fixed (Table 1). Therefore, the boundary conditions on the crack surfaces can be free–free, fixed–fixed or free–fixed. The special cases in which both materials are identical or one material is absent are also considered. After presenting the basic equations and the Stroh formalism in Sections 2 and 3, we consider in Sections 4 and 5 the special cases and in Section 6 the composite wedges. For simplicity in presentation, we assume that $\phi = 0$ in these sections and prove in Section 7 that δ's so obtained apply also to $\phi \neq 0$. Thus the order of singularities δ is invariant with respect to the orientation of the surface $\theta = \phi$. In all cases it is shown that if δ is a root so is $\delta + n$, where n is an integer. Therefore, it suffices

Table 1. The order of stress singularity ($\sigma \cong r^{\delta}$) for some geometry of anisotropic elastic materials and composites

Crack surfaces / Geometry	1. Free–free		2. Fixed–fixed		3. Free–fixed $\theta = \phi + \pi$ is free surface
 *Case I*	$\delta = 0$	(3)	$\delta = 0$	(3)	$\delta = -\frac{1}{2}$ $= -\frac{1}{2} \pm i\gamma$ γ given in (4.24)
 *Case II*	$\delta = 0$ $= -\frac{1}{2}$	(3) (3)	$\delta = 0$ $= -\frac{1}{2}$	(3) (3)	$\delta = -\frac{1}{4}$ $= -\frac{1}{4} \pm i\frac{\gamma}{2}$ $= -\frac{3}{4}$ $= -\frac{3}{4} \pm i\frac{\gamma}{2}$ γ given in (4.24)
 *Case III*	$\delta = 0,$ $= -\frac{1}{2}$ $= -\frac{1}{2} \pm i\gamma$ γ given in (6.16)	(3)	$\delta = 0,$ $= -\frac{1}{2}$ $= -\frac{1}{2} \pm i\gamma$ γ given in (6.27)	(3)	$e^{i2\delta\pi} = \frac{\lambda - 1}{\lambda + 1}$ λ = roots of (6.34) For isotropic composites, δ's are given in (6.37).

Note: If δ is a root, so is $\delta + n$ where n is an integer. Hence, only $-1 < \mathrm{Re}(\delta) \leqslant 0$ are given in the table. In case of a repeated root, the number in the parentheses after the δ indicates the multiplicity of the root. The value of δ is independent of the orientation ϕ of the crack, the boundary surface, or the interface.

to consider δ in the region $-1 < \mathrm{Re}(\delta) \leqslant 0$. Explicit solutions for δ's are obtained for all cases except Case III-3 for which only the solutions for isotropic materials are given. In all cases, the solutions associated with the degenerate cases of isotropic materials are deduced which agree with the published results in the literature.

2. BASIC EQUATIONS

In a fixed rectangular coordinate system (x_1, x_2, x_3), let the stress–strain law of an anisotropic elastic material be given by

$$\sigma_{ij} = c_{ijks}\varepsilon_{ks}, \tag{2.1}$$

where the repeated indices imply summation, σ_{ij} and ε_{ks} are, respectively, the stress and strain, and c_{ijks} are the elasticity constants having the symmetry properties

$$c_{ijks} = c_{jiks} = c_{ijsk} = c_{ksij}. \tag{2.2}$$

If u_i are the displacement components, the strain-displacement and equilibrium equations are

$$\varepsilon_{ks} = \tfrac{1}{2}(u_{k,s}+u_{s,k}), \tag{2.3}$$

$$\sigma_{ij,j} = 0, \tag{2.4}$$

in which a comma stands for partial differentiation. Use of (2.3) and (2.2) in (2.1) yields

$$\sigma_{ij} = c_{ijks}u_{k,s}. \tag{2.5}$$

Consider a two-dimensional deformation in which u_k $(k = 1, 2, 3)$ depends on x_1 and x_2 only. Assuming that

$$u_k = a_k f(Z), \tag{2.6a}$$

$$Z = x_1 + px_2, \tag{2.6b}$$

where p and a_k are, respectively, the eigenvalue and eigenvector of the elasticity constants to be determined, and f is an arbitrary function of Z, (2.5) and (2.4) yield

$$\sigma_{ij} = (c_{ijk1}+pc_{ijk2})a_k \frac{\mathrm{d}}{\mathrm{d}Z} f(Z), \tag{2.7}$$

$$\{c_{i1k1}+p(c_{i1k2}+c_{i2k1})+p^2c_{i2k2}\}a_k = 0. \tag{2.8}$$

Introducing the 3×3 matrices

$$\begin{aligned} Q_{ik} &= c_{i1k1}, \\ R_{ik} &= c_{i1k2}, \\ T_{ik} &= c_{i2k2}, \end{aligned} \tag{2.9}$$

eqn (2.8) can be rewritten in matrix notations as

$$\{\mathbf{Q}+p(\mathbf{R}+\mathbf{R}^T)+p^2\mathbf{T}\}\mathbf{a} = \mathbf{0}, \tag{2.10}$$

in which the superscript T stands for the transpose. Notice that $\mathbf{T}$ and $\mathbf{Q}$ are symmetric and positive definite if the strain energy is positive. For a nontrivial solution of $\mathbf{a}$, we must have

$$\|\mathbf{Q}+p(\mathbf{R}+\mathbf{R}^T)+p^2\mathbf{T}\| = 0, \tag{2.11}$$

from which p can be obtained. The eigenvector $\mathbf{a}$ is then determined from (2.10).

If we rewrite (2.10) as

$$(\mathbf{Q}+p\mathbf{R})\mathbf{a} = -p(\mathbf{R}^T+p\mathbf{T})\mathbf{a}, \tag{2.12}$$

and introduce a new vector

$$\begin{aligned}\mathbf{b} &= (\mathbf{R}^T+p\mathbf{T})\mathbf{a} \\ &= -(1/p)(\mathbf{Q}+p\mathbf{R})\mathbf{a},\end{aligned} \tag{2.13}$$

eqn (2.7) for $j = 1$ and 2 can be written as, using (2.9) and (2.13),

$$\begin{aligned}\sigma_{i1} &= -pb_i \frac{\mathrm{d}}{\mathrm{d}Z} f(Z), \\ \sigma_{i2} &= b_i \frac{\mathrm{d}}{\mathrm{d}Z} f(Z).\end{aligned} \tag{2.14}$$

In a polar coordinate system (r, θ) defined by

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \tag{2.15}$$

the complex variable Z becomes

$$Z = r(\cos\theta + p\sin\theta). \tag{2.16}$$

Let t_i be the surface traction vector on a radial plane which makes an angle θ with the x_1-axis. Since the unit normal to the radial plane is $n_i = (-\sin\theta, \cos\theta, 0)$, we have

$$t_i = \sigma_{ij}n_j = -\sigma_{i1}\sin\theta + \sigma_{i2}\cos\theta,$$

or, using (2.14) and (2.16),

$$\mathbf{t} = \frac{1}{r}\,\mathbf{b}Z\frac{\mathrm{d}}{\mathrm{d}Z} f(Z). \tag{2.17}$$

Thus $\mathbf{b}$ is proportional to $\mathbf{t}$ while $\mathbf{a}$ is proportional to $\mathbf{u}$. Notice that the complex variable Z is of order r according to (2.16).

3. THE SEXTIC FORMALISM OF STROH

The two equations in (2.13) can be written as an eigenvalue problem

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{3.1}$$

in which

$$\mathbf{N} = \begin{bmatrix} -\mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1} \\ \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} & -\mathbf{R}\mathbf{T}^{-1}\end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}. \tag{3.2}$$

Therefore p is a root of

$$\|\mathbf{N} - p\mathbf{I}\| = 0. \tag{3.3}$$

Equation (3.3) is an alternate form of (2.11). Notice that the left-hand side of (3.3) is a 6×6

determinant, while that of (2.11) is a 3×3 determinant. It can be shown[30] that p cannot be real if the strain energy is positive. We therefore have three pairs of complex conjugates for p. If p_ω $(\omega = 1, 2, \ldots, 6)$ are the roots, we let

$$p_{\omega+3} = \bar{p}_\omega \qquad (\omega = 1, 2, 3), \tag{3.4}$$

where an over bar denotes the complex conjugate. The eigenvectors $\boldsymbol{\xi}_\omega$ have the similar property

$$\boldsymbol{\xi}_{\omega+3} = \bar{\boldsymbol{\xi}}_\omega \qquad (\omega = 1, 2, 3). \tag{3.5}$$

Unless stated otherwise, we assume that p_ω are distinct. The degenerate case in which p_ω is a repeated eigenvalue is studied in [39].

Introducing the 6×6 matrix

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{J} = \mathbf{J}^T = \mathbf{J}^{-1}, \tag{3.6}$$

we have

$$\mathbf{JN} = \begin{bmatrix} \mathbf{RT}^{-1}\mathbf{R}^T - \mathbf{Q} & -\mathbf{RT}^{-1} \\ -\mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1} \end{bmatrix}. \tag{3.7}$$

Since $\mathbf{T}$ and $\mathbf{Q}$ are symmetric,

$$\mathbf{JN} = (\mathbf{JN})^T = \mathbf{N}^T\mathbf{J}. \tag{3.8}$$

Rewriting (3.1) as

$$(\mathbf{JN})\boldsymbol{\xi} = p\mathbf{J}\boldsymbol{\xi}, \tag{3.9}$$

in which (**JN**) and **J** are symmetric, it can be shown that[40]

$$\boldsymbol{\xi}_\omega^T \mathbf{J} \boldsymbol{\xi}_\psi = 0, \qquad \text{if} \quad \omega \neq \psi. \tag{3.10}$$

This orthogonality relation was first derived by Stroh[28]. In view of the fact that $\boldsymbol{\xi}_\omega$ is uniquely determined up to an arbitrary multiplicative constant, we can normalize the vector $\boldsymbol{\xi}_\omega$ by letting

$$\boldsymbol{\xi}_\omega^T \mathbf{J} \boldsymbol{\xi}_\psi = \delta_{\omega\psi}, \tag{3.11}$$

where $\delta_{\omega\psi}$ is the Kronecker delta. If we define the 3×3 matrices **A** and **B** by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \qquad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3], \tag{3.12}$$

and the 6×6 matrix **U** by

$$\begin{aligned} \mathbf{U} &= [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2, \bar{\boldsymbol{\xi}}_3] \\ &= \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix}, \end{aligned} \tag{3.13}$$

it follows from (3.11) that

$$\mathbf{U}^T\mathbf{J}\mathbf{U} = \mathbf{I}. \tag{3.14}$$

Performing the matrix multiplication on the left-hand side using the definition of **U** and **J** from (3.13) and (3.6), we obtain the following form for the orthogonality relations

$$\mathbf{A}^T\mathbf{B}+\mathbf{B}^T\mathbf{A} = \mathbf{I} = \bar{\mathbf{A}}^T\bar{\mathbf{B}}+\bar{\mathbf{B}}^T\bar{\mathbf{A}}, \tag{3.15a}$$

$$\mathbf{A}^T\bar{\mathbf{B}}+\mathbf{B}^T\bar{\mathbf{A}} = \mathbf{0} = \bar{\mathbf{B}}^T\mathbf{A}+\bar{\mathbf{A}}^T\mathbf{B}. \tag{3.15b}$$

On the other hand, (3.14) implies that $\mathbf{U}^T$ is the inverse of (**JU**), and hence the product of $\mathbf{U}^T$ and (**JU**) can be interchanged. Therefore,

$$\mathbf{JUU}^T = \mathbf{I}, \tag{3.16a}$$

or

$$\mathbf{UU}^T = \mathbf{J}. \tag{3.16b}$$

By performing the matrix multiplication on the left-hand side, we have the closure relations[29]

$$\mathbf{AA}^T+\bar{\mathbf{A}}\bar{\mathbf{A}}^T = \mathbf{0} = \mathbf{BB}^T+\bar{\mathbf{B}}\bar{\mathbf{B}}^T, \tag{3.17a}$$

$$\mathbf{BA}^T+\bar{\mathbf{B}}\bar{\mathbf{A}}^T = \mathbf{I} = \mathbf{AB}^T+\bar{\mathbf{A}}\bar{\mathbf{B}}^T. \tag{3.17b}$$

Equations (3.17a) imply that $\mathbf{AA}^T$ and $\mathbf{BB}^T$ are purely imaginary. Let

$$\mathbf{H} = 2i\mathbf{AA}^T = \mathbf{H}^T, \tag{3.18}$$

$$\mathbf{L} = -2i\mathbf{BB}^T = \mathbf{L}^T, \tag{3.19}$$

where **H** and **L** are real and symmetric matrices. Moreover, it can be shown that both **H** and **L** are positive definite if the strain energy is[39]. Consequently, (3.18) and (3.19) insure that **A** and **B** are nonsingular. Equations (3.17b) imply that

$$\mathbf{AB}^T = \tfrac{1}{2}(\mathbf{I}-i\mathbf{S}), \tag{3.20a}$$

$$\mathbf{S} = i(2\mathbf{AB}^T-\mathbf{I}), \tag{3.20b}$$

where **S** is real. Using (3.18)–(3.20) and the fact that $\mathbf{Ab}^{-1} = \mathbf{AB}^T(\mathbf{BB}^T)^{-1}$ and $\mathbf{BA}^{-1} = (\mathbf{AB}^T)^T(\mathbf{AA}^T)^{-1}$, we have

$$\mathbf{AB}^{-1} = -(\mathbf{S}+i\mathbf{I})\mathbf{L}^{-1} = \mathbf{L}^{-1}(\mathbf{S}^T-i\mathbf{I}), \tag{3.21}$$

$$\mathbf{BA}^{-1} = (\mathbf{S}^T+i\mathbf{I})\mathbf{H}^{-1} = -\mathbf{H}^{-1}(\mathbf{S}-i\mathbf{I}). \tag{3.22}$$

The second equality in (3.21) and (3.22) comes from the identities (3.23b) and (3.24b), which show below that $\mathbf{SL}^{-1}$ and $\mathbf{S}^T\mathbf{H}^{-1}$ are antisymmetric.

The real matrices **H**, **L** and **S** are not entirely independent. Indeed they are related by the following identities:

$$\mathbf{LS}+\mathbf{S}^T\mathbf{L} = \mathbf{0}, \tag{3.23a}$$

$$\mathbf{SL}^{-1}+\mathbf{L}^{-1}\mathbf{S}^T = \mathbf{0}, \tag{3.23b}$$

$$\mathbf{HS}^T+\mathbf{SH} = \mathbf{0}, \tag{3.24a}$$

$$\mathbf{S}^T\mathbf{H}^{-1}+\mathbf{H}^{-1}\mathbf{S} = \mathbf{0}, \tag{3.24b}$$

$$\mathbf{HL}-\mathbf{SS} = \mathbf{I}. \tag{3.25}$$

Identities (3.23a), (3.24a) and (3.25) can be verified by a direct substitution of **H**, **L** and **S** from (3.18) to (3.20) with the aid of (3.15a). Identity (3.23b) is obtained from (3.23a) by premultiplying and post-multiplying by $\mathbf{L}^{-1}$. Similarly, (3.24b) is obtained from (3.24a) by multiplying by $\mathbf{H}^{-1}$.

All equations derived so far, starting from (3.11), are based on the assumption that the eigenvalues p_ω's are distinct. In the degenerate cases in which p_ω is a repeated root, say of multiplicity 3, the derivation remains valid if the degeneracy is "semisimple", i.e. if three independent eigenvectors $\boldsymbol{\xi}_\omega$ associated with the repeated p_ω exist. If the degeneracy is "nonsemisimple", such as in the case of isotropic materials in which $p_\omega = i$ is a triple root but only two independent $\boldsymbol{\xi}_\omega$'s exist, one may introduce generalized eigenvectors[39] so that most of the equations derived remain valid. In many anisotropic elasticity problems including the one in this paper, the final solution requires only one or more of the three real matrices **H**, **L** and **S**, not p_ω, **A** or **B**, which are complex. In [36], Barnett and Lothe introduce an integral formalism to determine **H**, **L** and **S** directly from a generalized definition of **Q**, **R** and **T** of (2.9). Their approach eliminates the need of finding the eigenvalues p_ω and eigenvectors $\boldsymbol{\xi}_\omega$ so that the problem of repeated eigenvalues disappears. We list below the expression for **H**, **L** and **S** for isotropic materials[38]:

$$\mathbf{H} = \frac{1}{\mu m}\begin{bmatrix} m-1 & 0 & 0 \\ 0 & m-1 & 0 \\ 0 & 0 & m \end{bmatrix}, \tag{3.26}$$

$$\mathbf{L} = \frac{\mu}{m}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & m \end{bmatrix}, \tag{3.27}$$

$$\mathbf{S} = \frac{1}{m}\begin{bmatrix} 0 & -(m-2) & 0 \\ m-2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.28}$$

$$m = 4(1-\nu), \qquad 2 \leqslant m \leqslant 4. \tag{3.29}$$

In the above, μ and ν are, respectively, the shear modulus and Poisson's ratio. In the notations of [38] and [39], $\mathbf{H}, \mathbf{L}, \mathbf{S}$ are $-\mathbf{Q}, 4\pi\mathbf{B}, \mathbf{S}$ and $\mathbf{S}_2, -\mathbf{S}_3, \mathbf{S}_1$, respectively.

4. ELASTIC WEDGE OF WEDGE ANGLE π

To find the stress singularities at the apex of an anisotropic elastic wedge of any wedge angle, we choose

$$f(Z) = Z^{\delta+1}/(\delta+1) \tag{4.1}$$

in (2.6a), (2.7) and (2.17) and linearly superimpose solutions associated with the six p_ω's. For (2.6a) and (2.17), we have

$$\mathbf{u}(\theta) = \sum_{\omega=1}^{3} (q_\omega \mathbf{a}_\omega Z_\omega^{\delta+1} + h_\omega \bar{\mathbf{a}}_\omega \bar{Z}_\omega^{\delta+1})/(\delta+1), \tag{4.2}$$

$$\mathbf{t}(\theta) = \sum_{\omega=1}^{3} \frac{1}{r} (q_\omega \mathbf{b}_\omega Z_\omega^{\delta+1} + h_\omega \bar{\mathbf{b}}_\omega \bar{Z}_\omega^{\delta+1}), \tag{4.3}$$

in which q_ω and h_ω are arbitrary constants and

$$Z_\omega = x_1 + p_\omega x_2 = r(\cos\theta + p_\omega \sin\theta). \tag{4.4}$$

We see that with the assumption of (4.1) the stresses given by (2.7) as well as the surface traction vector $\mathbf{t}$ of (4.3) are of the order r^δ. The stresses are singular if the real part of δ is negative. For the strain energy to be bounded at the wedge apex, we require that $\mathrm{Re}(\delta) > -1$. If the wedge is bounded by $\theta_1 \leqslant \theta \leqslant \theta_2$ and the boundary conditions on the radial surfaces $\theta = \theta_1$ and θ_2 are either traction-free ($\mathbf{t} = \mathbf{0}$) or fixed ($\mathbf{u} = \mathbf{0}$), application of (4.2) and (4.3) yields a system of homogeneous equations for q_ω and h_ω. If $\mathbf{K}(\delta)$ is the coefficient matrix of q_ω and h_ω, a nontrivial solution of q_ω and h_ω demands that

$$\|\mathbf{K}(\delta)\| = 0, \tag{4.5}$$

which provides the desired δ. In applications, one is interested in δ's whose real part is negative as well as positive[23, 26, 27]. Thus one has to search numerically for the roots of (4.5) in the region $\mathrm{Re}(\delta) > -1$ of the complex plane δ. Notice that $\mathbf{K}(\delta)$ is a 6×6 matrix for the single wedge problem and a 12×12 matrix for the composite wedge that we will discuss in Section 6.

In this paper we will study special wedges and composite wedges whose wedge angle is either π or 2π. In all cases, the determinant (4.5) can be simplified substantially and an explicit solution for δ is obtained except for Case III-3. We will show that if δ is a root, so is $\delta+n$, where n is an integer. Hence we can focus our attention on $-1 < \mathrm{Re}(\delta) \leqslant 0$. We will also see that if the wedge boundaries are the radial planes $\theta = \phi$ and $\theta = \phi \pm \pi$, δ is independent of ϕ. For simplicity in presentation, however, we will present δ for $\phi = 0$ and leave the proof of invariance to Section 7. Noticing that

$$Z_\omega = \begin{cases} r, & \text{when } \theta = 0, \\ r\,\mathrm{e}^{\pm i\pi}, & \text{when } \theta = \pm\pi, \end{cases} \tag{4.6}$$

(4.2) and (4.3) yield

$$\mathbf{u}(0) = r^{\delta+1}(\mathbf{A}\mathbf{q}+\bar{\mathbf{A}}\mathbf{h})/(\delta+1) \tag{4.7}$$

$$\mathbf{t}(0) = r^{\delta}(\mathbf{B}\mathbf{q}+\bar{\mathbf{B}}\mathbf{h}) \tag{4.8}$$

$$\mathbf{u}(\pm\pi) = -r^{\delta+1}(\mathrm{e}^{\pm i\delta\pi}\mathbf{A}\mathbf{q}+\mathrm{e}^{\mp i\delta\pi}\bar{\mathbf{A}}\mathbf{h})/(\delta+1) \tag{4.9}$$

$$\mathbf{t}(\pm\pi) = -r^{\delta}(\mathrm{e}^{\pm i\delta\pi}\mathbf{B}\mathbf{q}+\mathrm{e}^{\mp i\delta\pi}\bar{\mathbf{B}}\mathbf{h}), \tag{4.10}$$

where $\mathbf{q}$ and $\mathbf{h}$ are 1×3 matrices whose elements are q_ω and h_ω ($\omega = 1,2,3$), respectively. In (4.9) and (4.10) we have made use of the property that $\mathrm{e}^{\pm i(\delta+1)\pi} = -\mathrm{e}^{\pm i\delta\pi}$.

Consider now an elastic wedge of wedge angle π which occupies the region $0 \leqslant \theta \leqslant \pi$. The radial planes $\theta = 0$ and $\theta = \pi$ can be traction free or fixed. We will discuss the three possible combinations of boundary conditions separately.

Case I-1. Free–free wedge

In this case, $\mathbf{t}(0) = \mathbf{t}(\pi) = \mathbf{0}$ and (4.8) and (4.10) yield

$$\mathbf{B}\mathbf{q}+\bar{\mathbf{B}}\mathbf{h} = \mathbf{0}, \tag{4.11}$$

$$\mathrm{e}^{i2\delta\pi}\mathbf{B}\mathbf{q}+\bar{\mathbf{B}}\mathbf{h} = \mathbf{0}. \tag{4.12}$$

Elimination of $\bar{\mathbf{B}}\mathbf{h}$ results in the equation

$$(1-\mathrm{e}^{i2\delta\pi})\mathbf{B}\mathbf{q} = \mathbf{0}. \tag{4.13}$$

Since $\mathbf{B}$ is nonsingular, for a nontrivial solution of $\mathbf{q}$ we must have

$$(1-\mathrm{e}^{i2\delta\pi})^3 = 0. \tag{4.14}$$

We see that if δ is a root, so is $\delta+n$, where n is any integer. For $-1 < \mathrm{Re}(\lambda) \leqslant 0$, the only root is $\delta = 0$. Moreover, $\delta = 0$ is a root of multiplicity three (Table 1).

Case I-2. Fixed–fixed wedge

In this case, $\mathbf{u}(0) = \mathbf{u}(\pi) = \mathbf{0}$ and (4.7) and (4.9) yield the same result as (4.14). Hence $\delta = 0$ is a triple root (Table 1). Notice that in this case as well as in Case I-1, the δ's are independent of material property. Hence the δ's remain the same if the half-plane boundary is at $\theta = \phi$ instead of $\theta = 0$ where ϕ is an arbitrary angle.

Case I-3. Free–fixed wedge

When $\mathbf{t}(\pi) = \mathbf{u}(0) = \mathbf{0}$, we have (4.12) and

$$\mathbf{Aq}+\bar{\mathbf{A}}\mathbf{h} = \mathbf{0}. \tag{4.15}$$

If we solve for $\mathbf{h}$ from (4.12),

$$\mathbf{h} = -e^{i2\delta\pi}\bar{\mathbf{B}}^{-1}\mathbf{Bq}, \tag{4.16}$$

and substitute into (4.15), we have

$$(\mathbf{AB}^{-1}-e^{i2\delta\pi}\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1})\mathbf{Bq} = \mathbf{0}. \tag{4.17}$$

For a nontrivial solution of $\mathbf{q}$, we have

$$\|\mathbf{AB}^{-1}-e^{i2\delta\pi}\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1}\| = 0, \tag{4.18}$$

or, using (3.21),

$$\|(1-e^{i2\delta\pi})\mathbf{SL}^{-1}+i(1+e^{i2\delta\pi})\mathbf{L}^{-1}\| = 0. \tag{4.19}$$

We see that if δ is a root, so is $\delta+n$ where n is an integer. Since $\mathbf{L}^{-1}$ is positive definite, the determinant is nonzero when we set $\delta = 0$. Hence $(1-e^{i2\delta\pi}) \neq 0$ and we may write (4.19) as

$$\|\mathbf{S}+i\lambda\mathbf{I}\| = 0, \tag{4.20}$$

in which

$$\lambda = \frac{1+e^{i2\delta\pi}}{1-e^{i2\delta\pi}}, \tag{4.21a}$$

or

$$e^{i2\delta\pi} = -\frac{1-\lambda}{1+\lambda}. \tag{4.21b}$$

By writing (4.20) as

$$\|\mathbf{SL}^{-1}+i\lambda\mathbf{L}^{-1}\| = 0,$$

where $(\mathbf{SL}^{-1})$ is antisymmetric according to (3.23b), the theorem proved in the Appendix applies here. If we identify $(\mathbf{SL}^{-1})$ and $\mathbf{L}^{-1}$ as $\mathbf{W}$ and $\mathbf{D}$ in eqn (A1), eqn (A2) becomes

$$-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{S}^2) > 0, \tag{4.22}$$

and (A3) gives

$$\lambda = 0, \qquad \lambda = \pm[-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{S}^2)]^{1/2}. \tag{4.23}$$

With λ given by (4.23), (4.21b) yields (Table 1)

$$\delta = -\tfrac{1}{2}, \qquad \delta = -\tfrac{1}{2} \pm i\gamma,$$
$$\gamma = \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta}, \qquad \beta = [-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{S}^2)]^{1/2}. \tag{4.24}$$

Notice that γ depends on the material property. If the half-plane boundary were at $\theta = \phi$ instead of $\theta = 0$, we could use a new coordinate system (x_1^*, x_2^*, x_3^*) in which the x_1^*-axis is the half-plane boundary. The material constants referred to this new coordinate system would be different and so would γ. However, we will show in Section 7 that γ and hence δ are invariant with respect to the orientation of the half-plane boundary.

For isotropic materials, $\mathbf{S}$ is given by (3.28) and

$$\beta = (m-2)/m < 1. \tag{4.25}$$

Hence γ of (4.24) is real and given by

$$\gamma = (1/2\pi)\ln(m-1). \tag{4.26}$$

This agrees with the result obtained in [2] for an isotropic wedge of wedge angle π. Notice that the singularity $\delta = \frac{1}{2}$ in (4.24) is, for isotropic materials, associated with the out-of-plane displacement while $\delta = -\frac{1}{2} \pm i\gamma$ are associated with in-plane displacements.

5. ELASTIC WEDGE OF WEDGE ANGLE 2π (A CRACK)

In this section we consider a crack along the negative x_1-axis. Thus the material occupies the region $-\pi \leqslant \theta \leqslant \pi$. Again, there are three possible combinations of the boundary conditions at $\theta = \pm\pi$. We will see that if δ is a root for Case I, $\delta/2$ is a root for Case II.

Case II-1. Free–free wedge

In this case, $\mathbf{t}(\pi) = \mathbf{t}(-\pi) = \mathbf{0}$ and (4.10) yields (4.12) and

$$\mathbf{Bq} + e^{i2\delta\pi}\bar{\mathbf{B}}\mathbf{h} = \mathbf{0}. \tag{5.1}$$

Elimination of $\bar{\mathbf{B}}\mathbf{h}$ between (4.12) and (5.1) leads to

$$(1 - e^{i4\delta\pi})\mathbf{Bq} = \mathbf{0}. \tag{5.2}$$

This is identical to (4.13) if we replace δ in (5.2) by $\delta/2$. Thus the values of δ for this case are one-half of the values of δ in Case I-1. To obtain δ from Case I-1, we should extend the range of δ to $-2 < \mathrm{Re}(\delta) \leqslant 0$ in Case I-1. Now since $\delta = -1$ and 0 are roots for Case I-1, $\delta = -\frac{1}{2}$ and 0 are the roots in the range $-1 < \mathrm{Re}(\delta) \leqslant 0$ here. Moreover, the roots are of multiplicity 3 (Table 1).

Case II-2. Fixed–fixed wedge

We have $\mathbf{u}(\pi) = \mathbf{u}(-\pi) = \mathbf{0}$ for this case, and it is not difficult to see that application of (4.9) yields the same δ as in Case II-1.

Case II-3. Free–fixed wedge

When $\mathbf{t}(\pi) = \mathbf{u}(-\pi) = \mathbf{0}$, the equations are given by (4.12) or (4.16) and

$$\mathbf{Aq} + e^{i2\delta\pi}\bar{\mathbf{A}}\mathbf{h} = \mathbf{0}. \tag{5.3}$$

If we substitute $\mathbf{h}$ from (4.16), we obtain

$$(\mathbf{A}\mathbf{B}^{-1} - e^{i4\delta\pi}\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1})\mathbf{B}\mathbf{q} = \mathbf{0}. \tag{5.4}$$

This is identical to (4.17) except a factor of 2 for the δ. Hence the solution for δ can be deduced from (4.24) as

$$\delta = -\tfrac{1}{4}, \qquad \delta = -\tfrac{1}{4} \pm i(\gamma/2), \tag{5.5a}$$

where γ is defined in (4.24). In (4.24) $\delta - 1$ is also a root and one-half of $\delta - 1$ in (4.24) yields

$$\delta = -\tfrac{3}{4}, \qquad \delta = -\tfrac{3}{4} \pm i(\gamma/2). \tag{5.5b}$$

Thus there are six stress singularities for this case versus three for Case I-3 (Table 1).

The solution for the special case of isotropic materials[2] is obtained when γ of (4.26) is used in (5.5). For isotropic materials, $\delta = -\tfrac{1}{4}$ and $-\tfrac{3}{4}$ are the singularities associated with out-of-plane displacement, while $\delta = -\tfrac{1}{4} \pm i(\gamma/2)$ and $-\tfrac{3}{4} \pm i(\gamma/2)$ are associated with the in-plane displacements.

6. INTERFACE CRACK

Let the x_1-axis be the interface between two anisotropic elastic materials and let the negative x_1-axis be the interface crack. Thus one material occupies the region $0 \leqslant \theta \leqslant \pi$ while the other material $-\pi \leqslant \theta \leqslant 0$. We will use a superscript prime to indicate those quantities associated with the material in $-\pi \leqslant \theta \leqslant 0$. The continuity of displacements and surface traction across the interface at $\theta = 0$ imply that $\mathbf{u}(0) = \mathbf{u}'(0)$ and $\mathbf{t}(0) = \mathbf{t}'(0)$, or by (4.7) and (4.8),

$$\mathbf{A}\mathbf{q} + \bar{\mathbf{A}}\mathbf{h} = \mathbf{A}'\mathbf{q}' + \bar{\mathbf{A}}'\mathbf{h}', \tag{6.1}$$

$$\mathbf{B}\mathbf{q} + \bar{\mathbf{B}}\mathbf{h} = \mathbf{B}'\mathbf{q}' + \bar{\mathbf{B}}'\mathbf{h}'. \tag{6.2}$$

We now consider various boundary conditions at $\theta = \pm\pi$.

Case III-1. Free–free crack

When $\mathbf{t}(\pi) = \mathbf{t}'(-\pi) = \mathbf{0}$, we have from (4.10)

$$e^{i2\delta\pi}\mathbf{B}\mathbf{q} + \bar{\mathbf{B}}\mathbf{h} = \mathbf{0}, \tag{6.3}$$

$$\mathbf{B}'\mathbf{q}' + e^{i2\delta\pi}\bar{\mathbf{B}}'\mathbf{h}' = \mathbf{0}, \tag{6.4}$$

or

$$\mathbf{h} = -e^{i2\delta\pi}\bar{\mathbf{B}}^{-1}\mathbf{B}\mathbf{q}, \tag{6.5}$$

$$\mathbf{q}' = -e^{i2\delta\pi}\mathbf{B}'^{-1}\bar{\mathbf{B}}'\mathbf{h}'. \tag{6.6}$$

Equations (6.1) and (6.2) can now be written as

$$(\mathbf{A}\mathbf{B}^{-1} - e^{i2\delta\pi}\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1})\mathbf{B}\mathbf{q} = (\bar{\mathbf{A}}'\bar{\mathbf{B}}'^{-1} - e^{i2\delta\pi}\mathbf{A}'\mathbf{B}'^{-1})\bar{\mathbf{B}}'\mathbf{h}', \tag{6.7}$$

$$(1 - e^{i2\delta\pi})\mathbf{B}\mathbf{q} = (1 - e^{i2\delta\pi})\bar{\mathbf{B}}'\mathbf{h}'. \tag{6.8}$$

The vanishing of the determinant of the coefficient matrix of $(\mathbf{B}\mathbf{q}, \bar{\mathbf{B}}'\mathbf{h}')$ reduces to either

$$(1 - e^{i2\delta\pi})^3 = 0, \tag{6.9}$$

or

$$\|\mathbf{A}\mathbf{B}^{-1} - \bar{\mathbf{A}}'\bar{\mathbf{B}}'^{-1} - e^{i2\delta\pi}(\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1} - \mathbf{A}'\mathbf{B}'^{-1})\| = 0. \tag{6.10}$$

We see that if δ is a root, so is $\delta+n$, where n is an integer. As before, we limit our attention to $-1 < \mathrm{Re}(\delta) \leqslant 0$. Equation (6.9) implies that $\delta = 0$ is a root of multiplicity 3 (Table 1). We will next study the roots of (6.10).

Using the expressions for $\mathbf{AB}^{-1}$ from (3.21), we obtain

$$\begin{aligned} \mathbf{AB}^{-1}-\bar{\mathbf{A}}'\bar{\mathbf{B}}'^{-1} &= -(\mathbf{W}+i\mathbf{D}),\\ \bar{\mathbf{A}}\bar{\mathbf{B}}^{-1}-\mathbf{A}'\mathbf{B}'^{-1} &= -(\mathbf{W}-i\mathbf{D}), \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} \mathbf{W} &= \mathbf{SL}^{-1}-\mathbf{S}'\mathbf{L}'^{-1},\\ \mathbf{D} &= \mathbf{L}^{-1}+\mathbf{L}'^{-1}. \end{aligned} \tag{6.12}$$

Notice that $\mathbf{D}$ is symmetric, positive definite and $\mathbf{W}$ is antisymmetric. Equation (6.10) can be rewritten in terms of real matrices $\mathbf{W}$ and $\mathbf{D}$ as

$$\|(1-e^{i2\delta\pi})\mathbf{W}+i(1+e^{i2\delta\pi})\mathbf{D}\| = 0. \tag{6.13}$$

Since $\delta = 0$ is not a root of (6.13), $(1-e^{i2\delta\pi}) \neq 0$, and we may write

$$\|\mathbf{W}+i\lambda\mathbf{D}\| = 0, \tag{6.14}$$

where λ is defined in (4.21a). It is shown in the Appendix that the three roots of (6.14) are all real and are given by

$$\begin{aligned} &\lambda = 0, \qquad \lambda = \pm\beta,\\ &\beta = [-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{WD}^{-1})^2]^{1/2}. \end{aligned} \tag{6.15}$$

With λ given by (6.15), δ of (4.21b) can be written as (Table 1)

$$\begin{aligned} &\delta = -\tfrac{1}{2}, \qquad \delta = -\tfrac{1}{2}\pm i\gamma,\\ &\gamma = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}, \qquad \beta = [-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{WD}^{-1})^2]^{1/2}. \end{aligned} \tag{6.16}$$

Equation (6.16) gives an explicit expression for the δ's. The fact that the singularities at an interface crack tip of an anisotropic composite can be written as $\delta = -\frac{1}{2}\pm i\gamma$ was stated in [24], but no proof was given. Nor was the expression for γ given except for a cross-ply composite.

For isotropic materials, $\delta = -\frac{1}{2}$ is associated with deformations due to antiplane displacements and $\delta = -\frac{1}{2}\pm i\gamma$ are associated with that due to in-plane displacements. The latter has been obtained in [3] and reproduced in [8] in a compact form. Using $\mathbf{L}$ and $\mathbf{S}$ in (3.27) and (3.28) for isotropic materials, (6.12) and (6.16) yield

$$\beta = \frac{\mu'(1-2\nu)-\mu(1-2\nu')}{2[\mu'(1-\nu)+\mu(1-\nu')]}. \tag{6.17}$$

With β given by (6.17), γ of (6.16) agrees with that obtained in [8]. Notice that β given by (6.17) is one of the two nondimensional parameters for isotropic composites introduced by Dundurs[41].

Case III-2. Fixed–fixed crack

With $\mathbf{u}(\pi) = \mathbf{u}'(-\pi) = \mathbf{0}$, (4.9) reduces to

$$\mathbf{h} = -e^{i2\delta\pi}\bar{\mathbf{A}}^{-1}\mathbf{Aq}, \tag{6.18}$$

$$\mathbf{q}' = -e^{i2\delta\pi}\mathbf{A}'^{-1}\bar{\mathbf{A}}'\mathbf{h}', \tag{6.19}$$

and (6.1) and (6.2) can be written as

$$(1-e^{i2\delta\pi})\mathbf{A}\mathbf{q} = (1-e^{i2\delta\pi})\bar{\mathbf{A}}'\mathbf{h}', \tag{6.20}$$

$$(\mathbf{B}\mathbf{A}^{-1}-e^{i2\delta\pi}\bar{\mathbf{B}}\bar{\mathbf{A}}^{-1})\mathbf{A}\mathbf{q} = (\bar{\mathbf{B}}'\bar{\mathbf{A}}'^{-1}-e^{i2\delta\pi}\mathbf{B}'\mathbf{A}'^{-1})\bar{\mathbf{A}}'\mathbf{h}'. \tag{6.21}$$

For a nontrivial solution of $\mathbf{Aq}$ and $\bar{\mathbf{A}}'\mathbf{h}'$, we must have (6.9) which gives $\delta = 0$ as a triple root or

$$\|\mathbf{B}\mathbf{A}^{-1}-\bar{\mathbf{B}}'\bar{\mathbf{A}}'^{-1}-e^{i2\delta\pi}(\bar{\mathbf{B}}\bar{\mathbf{A}}^{-1}-\mathbf{B}'\mathbf{A}'^{-1})\| = 0. \tag{6.22}$$

Using the expression for $\mathbf{BA}^{-1}$ from (3.22), we obtain

$$\begin{aligned}\mathbf{B}\mathbf{A}^{-1}-\bar{\mathbf{B}}'\bar{\mathbf{A}}'^{-1} &= \hat{\mathbf{W}}+i\hat{\mathbf{D}},\\ \bar{\mathbf{B}}\bar{\mathbf{A}}^{-1}-\mathbf{B}'\mathbf{A}'^{-1} &= \hat{\mathbf{W}}-i\hat{\mathbf{D}},\end{aligned} \tag{6.23}$$

where

$$\begin{aligned}\hat{\mathbf{W}} &= -(\mathbf{H}^{-1}\mathbf{S}-\mathbf{H}'^{-1}\mathbf{S}'),\\ \hat{\mathbf{D}} &= \mathbf{H}^{-1}+\mathbf{H}'^{-1}.\end{aligned} \tag{6.24}$$

Thus $\hat{\mathbf{W}}$ is real and antisymmetric, while $\hat{\mathbf{D}}$ is real, symmetric and positive definite. By substituting (6.23) into (6.22) and observing that $\delta = 0$ is not a root of the resulting determinant, we have

$$\|\hat{\mathbf{W}}+i\lambda\hat{\mathbf{D}}\| = 0, \tag{6.25}$$

in which λ is defined in (4.21a). Equation (6.25) is similar to (6.14), and hence the roots λ are all real (see the Appendix):

$$\begin{aligned}&\lambda = 0, \qquad \lambda = \pm\beta,\\ &\beta = [-\tfrac{1}{2}\,\mathrm{tr}(\hat{\mathbf{W}}\hat{\mathbf{D}}^{-1})^2]^{1/2}.\end{aligned} \tag{6.26}$$

In terms of δ given by (4.21b), the roots are

$$\begin{aligned}&\delta = -\tfrac{1}{2}, \qquad \delta = -\tfrac{1}{2}\pm i\gamma,\\ &\gamma = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}, \qquad \beta = [-\tfrac{1}{2}\,\mathrm{tr}(\hat{\mathbf{W}}\hat{\mathbf{D}}^{-1})^2]^{1/2}.\end{aligned} \tag{6.27}$$

The associated problem for the special case of isotropic material has been investigated by Erdogan and Gupta[10]. In [42], composite wedges of arbitrary wedge angles are studied. If we use (3.26) and (3.28) in (6.24) and then (6.27), we have

$$\beta = \frac{\mu(m-2)(m'-1)-\mu'(m'-2)(m-1)}{\mu m(m'-1)+\mu' m'(m-1)}. \tag{6.28}$$

With β given by (6.28) for isotropic composites, γ in (6.27) reproduces the result obtained in [10]. As in Case III-1, $\delta = -\tfrac{1}{2}$ in (6.27) is associated with the antiplane displacements, while $\delta = -\tfrac{1}{2}\pm i\gamma$ are associated with in-plane displacements for isotropic materials.

Case III-3. Free–fixed crack

Let $\mathbf{t}(\pi) = \mathbf{u}'(-\pi) = \mathbf{0}$. Using (6.5) and (6.19) in (6.1) and (6.2), we have

$$(\mathbf{AB}^{-1} - e^{i2\delta\pi}\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1})\mathbf{Bq} = (1 - e^{i2\delta\pi})\bar{\mathbf{A}}'\bar{\mathbf{h}}' \tag{6.29}$$

$$(1 - e^{i2\delta\pi})\mathbf{Bq} = (\bar{\mathbf{B}}'\bar{\mathbf{A}}'^{-1} - e^{i2\delta\pi}\mathbf{B}'\mathbf{A}'^{-1})\bar{\mathbf{A}}'\bar{\mathbf{h}}'. \tag{6.30}$$

We see that if δ is a root, so is $\delta + n$, where n is an integer. By making use of (3.21) and (3.22) for $\mathbf{AB}^{-1}$ and $\mathbf{B}'\mathbf{A}'^{-1}$, and observing that $\delta = 0$ is not a root of the resulting equations, we have

$$(\mathbf{S} + i\lambda\mathbf{I})\mathbf{L}^{-1}\mathbf{Bq} = -\bar{\mathbf{A}}'\bar{\mathbf{h}}', \tag{6.31}$$

$$-\mathbf{Bq} = \mathbf{H}'^{-1}(\mathbf{S}' + i\lambda\mathbf{I})\bar{\mathbf{A}}'\bar{\mathbf{h}}', \tag{6.32}$$

where λ is defined in (4.21a). Substitution of $\bar{\mathbf{A}}'\bar{\mathbf{h}}'$ from (6.31) into (6.32) yields

$$\{\mathbf{H}'^{-1}(\mathbf{S}' + i\lambda\mathbf{I})(\mathbf{S} + i\lambda\mathbf{I})\mathbf{L}^{-1} - \mathbf{I}\}\mathbf{Bq} = \mathbf{0}. \tag{6.33}$$

For a nontrivial solution of $\mathbf{Bq}$, we must have

$$\|(\mathbf{S}' + i\lambda\mathbf{I})(\mathbf{S} + i\lambda\mathbf{I}) - \mathbf{H}'\mathbf{L}\| = 0,$$

or

$$\|(\mathbf{S}'\mathbf{S} - \mathbf{H}'\mathbf{L}) + i\lambda(\mathbf{S} + \mathbf{S}') + (i\lambda)^2\mathbf{I}\| = 0. \tag{6.34}$$

Equation (6.34) provides six roots for λ. δ is then obtained from (4.21b).

The associated problem for isotropic composites does not seem to have been studied. If we use $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}$ for isotropic materials from (3.26) to (3.28) in (6.34), the six roots are

$$\begin{aligned} &\lambda = \pm i(\mu/\mu')^{1/2}, \quad \text{and} \\ &\lambda = \pm\rho + i\eta, \quad \text{and} \quad \pm\rho - i\eta, \quad \text{if} \quad \mu/\mu' \geqslant \Delta, \\ &\lambda = \pm(\rho + \eta), \quad \text{and} \quad \pm(\rho - \eta), \quad \text{if} \quad \mu/\mu' < \Delta, \end{aligned} \tag{6.35}$$

where

$$\begin{aligned} \Delta &= \frac{(m - m')^2}{4mm'(m' - 1)}, \\ \rho &= 1 - (1/m + 1/m'), \\ \eta &= 2\left(\frac{m' - 1}{mm'}\left|\frac{\mu}{\mu'} - \Delta\right|\right)^{1/2}. \end{aligned} \tag{6.36}$$

the corresponding δ's are

$$\begin{aligned} &\delta = -\frac{1}{2} \mp \frac{1}{\pi}\tan^{-1}\left(\frac{\mu}{\mu'}\right)^{1/2}, \\ &\delta = -\tfrac{1}{2} - \alpha \pm i\beta, \quad \text{and} \quad -\tfrac{1}{2} + \alpha \pm i\beta, \quad \text{if} \quad \mu/\mu' \geqslant \Delta, \\ &\delta = -\tfrac{1}{2} \pm i\gamma_1, \quad \text{and} \quad -\tfrac{1}{2} \pm i\gamma_2, \quad \text{if} \quad \mu/\mu' < \Delta, \end{aligned} \tag{6.37}$$

where

$$\alpha = \frac{1}{2\pi}\tan^{-1}\left(\frac{2\eta}{1-\rho^2-\eta^2}\right),$$

$$\beta = \frac{1}{4\pi}\ln\{[(1+\rho)^2+\eta^2]/[(1-\rho)^2+\eta^2]\}, \tag{6.38}$$

$$\gamma_1 = \frac{1}{2\pi}\ln\frac{1+\rho+\eta}{1-\rho-\eta}, \qquad \gamma_2 = \frac{1}{2\pi}\ln\frac{1+\rho-\eta}{1-\rho+\eta}.$$

It should be pointed out that when $\mu/\mu' = \Delta$, (6.36) yields $\eta = 0$ and (6.38) provides $\alpha = 0$, $\beta = \gamma_1 = \gamma_2$. Thus $\delta = -\frac{1}{2} \pm i\beta$ are double roots, and we may have additional stress singularities of the form $r^\delta(\ln r)$[43].

To see how this case reduces to Case II-3 when the two materials are identical, we rewrite (6.34) as

$$\|\mathbf{M} + i2\lambda\bar{\mathbf{S}} - (\lambda^2+1)\mathbf{I}\| = 0, \tag{6.39}$$

in which

$$\bar{\mathbf{S}} = \tfrac{1}{2}(\mathbf{S}+\mathbf{S}'), \tag{6.40}$$

$$\mathbf{M} = \mathbf{S}'\mathbf{S} - \mathbf{H}'\mathbf{L} + \mathbf{I}. \tag{6.41}$$

If $\mathbf{M} = \mathbf{0}$, which is the case when the two materials are identical according to (3.25), we have

$$\|\bar{\mathbf{S}} + i\bar{\lambda}\mathbf{I}\| = 0, \tag{6.42}$$

$$\bar{\lambda} = \frac{\lambda^2+1}{2\lambda} = \frac{1+e^{i4\delta\pi}}{1-e^{i4\delta\pi}}. \tag{6.43}$$

These are identical to (4.20) and (4.21a), respectively, except a factor of 2 for δ. This agrees with the results stated in Case II-3. Notice that (6.42) and (6.43) remain valid even if the two materials in the composite are different as long as $\mathbf{M} = \mathbf{0}$. It would be interesting to study if there exists such an anisotropic composite for which $\mathbf{M} = \mathbf{0}$. For isotropic composites, $\mathbf{M} = \mathbf{0}$ only if the two isotropic materials are identical.

7. INVARIANCE OF δ WITH THE INTERFACE BOUNDARY

For the half-plane problems considered in Section 4, we could have considered the boundary of the half-plane to be at $\theta = \phi$ instead of $\theta = 0$. Likewise, the crack in Section 5 could be located along $\theta = \phi \pm \pi$ and in Section 6 the crack and interface could be located along $\theta = \phi \pm \pi$ and ϕ, respectively. We will show in this section that δ so obtained is independent of ϕ and hence identical to the δ for $\phi = 0$ considered in the previous sections.

To this end, we choose a new coordinate system (x_1^*, x_2^*, x_3^*) which is related to the original system (x_1, x_2, x_3) by

$$x_i^* = \Omega_{ij}x_j, \tag{7.1}$$

$$\mathbf{\Omega} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7.2}$$

so that $x_2^* = 0$ is the boundary of the half-plane, the crack or the interface. The formulations and solutions in Sections 4 to 6 apply here if a superscript $*$ is added to all quantities. For instance, (4.18) for Case I-3 becomes

$$\|\mathbf{A}^*\mathbf{B}^{*-1} - e^{i2\delta^*\pi}\bar{\mathbf{A}}^*\bar{\mathbf{B}}^{*-1}\| = 0. \tag{7.3}$$

It remains to prove that $\delta^* = \delta$. It is shown in [35] that under a change of coordinate systems the Stroh eigenvectors $\mathbf{a}$ and $\mathbf{b}$ of (2.10) and (2.13), albeit complex valued, transform according to the laws of transformation for tensors of order one. That is,

$$\mathbf{a}^* = \mathbf{\Omega a}, \qquad \mathbf{b}^* = \mathbf{\Omega b}. \tag{7.4}$$

It follows from (3.12) that

$$\mathbf{A}^* = \mathbf{\Omega A}, \qquad \mathbf{B}^* = \mathbf{\Omega B}, \tag{7.5}$$

and (7.3) reduces to

$$\|\mathbf{\Omega}(\mathbf{AB}^{-1} - e^{i2\delta^*\pi}\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1})\mathbf{\Omega}^T\| = 0, \tag{7.6}$$

or since $\|\mathbf{\Omega}\| = 1$,

$$\|\mathbf{AB}^{-1} - e^{i2\delta^*\pi}\bar{\mathbf{A}}\bar{\mathbf{B}}^{-1}\| = 0. \tag{7.7}$$

Compared with (4.18), it is clear that $\delta^* = \delta$. Other cases can be proved in the same way.

A direct proof of the invariance without resorting to the coordinate transformation can be made as follows. Noticing that Z_ω of (4.4) for $\theta = \phi$ and $\phi \pm \pi$ are related by (see also Fig. 1 of [34])

$$Z_\omega(\phi \pm \pi) = e^{\pm i\pi} Z_\omega(\phi), \tag{7.8}$$

and writing $Z_\omega(\phi)$ as

$$Z_\omega(\phi) = r\zeta_\omega(\phi), \qquad \zeta_\omega(\phi) = \cos\phi + p_\omega \sin\phi, \tag{7.9}$$

(4.3) for $\theta = \phi$ and $\phi \pm \pi$ become

$$\mathbf{t}(\phi) = \sum_{\omega=1}^{3} r^\delta\{q_\omega \mathbf{b}_\omega \zeta_\omega^{\delta+1}(\phi) + h_\omega \bar{\mathbf{b}}_\omega \bar{\zeta}_\omega^{\delta+1}(\phi)\}, \tag{7.10}$$

$$\mathbf{t}(\phi \pm \pi) = \sum_{\omega=1}^{3} r^\delta\{e^{\pm i(\delta+1)\pi} q_\omega \mathbf{b}_\omega \zeta_\omega^{\delta+1}(\phi) + e^{\mp i(\delta+1)\pi} h_\omega \bar{\mathbf{b}}_\omega \bar{\zeta}_\omega^{\delta+1}(\phi)\}. \tag{7.11}$$

Similar equations can be written for $\mathbf{u}(\phi)$ and $\mathbf{u}(\phi \pm \pi)$ of (4.2). Introducing the new coefficients

$$\begin{aligned} \hat{q}_\omega &= q_\omega \zeta_\omega^{\delta+1}(\phi) \qquad (\omega \text{ not summed}), \\ \hat{h}_\omega &= h_\omega \bar{\zeta}_\omega^{\delta+1}(\phi) \qquad (\omega \text{ not summed}), \end{aligned} \tag{7.12}$$

and noticing that $e^{\pm i(\delta+1)\pi} = -e^{\pm i\delta\pi}$, we have

$$\mathbf{u}(\phi) = r^{\delta+1}(\mathbf{A}\hat{\mathbf{q}} + \bar{\mathbf{A}}\hat{\mathbf{h}})/(\delta+1), \tag{7.13}$$

$$\mathbf{t}(\phi) = r^\delta(\mathbf{B}\hat{\mathbf{q}} + \bar{\mathbf{B}}\hat{\mathbf{h}}), \tag{7.14}$$

$$\mathbf{u}(\phi\pm\pi) = -r^{\delta+1}(e^{\pm i\delta\pi}\mathbf{A}\hat{\mathbf{q}}+e^{\mp i\delta\pi}\bar{\mathbf{A}}\hat{\mathbf{h}})/(\delta+1), \tag{7.15}$$

$$\mathbf{t}(\phi\pm\pi) = -r^{\delta}(e^{\pm i\delta\pi}\mathbf{B}\hat{\mathbf{q}}+e^{\mp i\delta\pi}\bar{\mathbf{B}}\hat{\mathbf{h}}). \tag{7.16}$$

Equations (7.13)–(7.16) are identical to (4.7)–(4.10) except that the undetermined coefficients are now $\hat{\mathbf{q}}$ and $\hat{\mathbf{h}}$ instead of $\mathbf{q}$ and $\mathbf{h}$. Therefore, δ does not depend on ϕ. The solutions for displacements and stresses however do depend on ϕ because $\hat{\mathbf{q}}$ and $\hat{\mathbf{h}}$, by virtue of (7.12), depend on ϕ.

The only work which we are aware of and which has some relevance to the above invariance property is the one by Barnett and Lothe[44]. They show that for dislocations in anisotropic bicrystals, certain prelogarithmic energy factors are independent of the orientation of the interface relative to Burger's vector in the bicrystal.

8. CONCLUDING REMARKS

When the order of stress singularity δ is a complex number and the crack surface is stress-free, the displacement becomes oscillatory near $r = 0$. For the case of a crack with free–free or free–fixed surface, the two crack surfaces penetrate each other. This is a physically unacceptable phenomenon, although the region of penetration is rather small[4, 10]. There have been several studies on the problem to eliminate the unrealistic oscillatory phenomenon[45–48]. We do not address to the problem in this paper. The purpose of this paper is to illustrate Stroh's powerful and elegant formalism in obtaining explicit solutions to a special problem. One possible way of alleviating the oscillatory phenomenon is to introduce a contact zone near the crack tip[45]. The stress singularity at the end of the contact zone has been analyzed in [49] for isotropic composites and in [50] for anisotropic composites using the present formulation.

It should be pointed out that not all composites with free–free crack surfaces have the interpenetration problem. By (6.16), the δ's are real if $\beta = 0$ or if $\mathbf{W} = \mathbf{0}$. It follows from (6.12) that there is no interpenetration problem if $\mathbf{SL}^{-1}$ for the two anisotropic materials in the composite are identical. Since $\mathbf{SL}^{-1}$ is antisymmetric according to (3.23b), $\mathbf{W} = \mathbf{0}$ leads to at most three conditions for the elasticity constants to be satisfied if there is to be no interpenetration problem. For isotropic composites, use of (3.27) and (3.28) leads to only one condition; namely, the value of $(1-2\nu)/\mu$ for the two materials to be identical.

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APPENDIX

Theorem: Let λ be a ropot of the 3×3 determinant

$$\|\mathbf{W}+i\lambda\mathbf{D}\| = 0, \tag{A1}$$

where **D** is a real, symmetric and positive definite matrix, while **W** is a real antisymmetric matrix. Then

$$-\tfrac{1}{2}\,\mathrm{tr}(\mathbf{W}\mathbf{D}^{-1})^2 > 0, \tag{A2}$$

and the three roots are all real given by

$$\lambda = 0, \qquad \lambda = \pm[-\tfrac{1}{2}\,\mathrm{tr}(\mathbf{W}\mathbf{D}^{-1})^2]^{1/2}. \tag{A3}$$

Proof: Equation (A1) is equivalent to

$$\|\mathbf{W}\mathbf{D}^{-1}+i\lambda\mathbf{I}\| = 0. \tag{A4}$$

Let I_1, I_2, I_3 be the principal invariants of $\mathbf{W}\mathbf{D}^{-1}$. Since **W** is antisymmetric, $\|\mathbf{W}\| = 0$ and

$$I_3 = \|\mathbf{W}\mathbf{D}^{-1}\| = \|\mathbf{W}\|\cdot\|\mathbf{D}^{-1}\| = 0. \tag{A5}$$

Also,

$$\begin{aligned} I_1 &= \mathrm{tr}(\mathbf{W}\mathbf{D}^{-1}) = \mathrm{tr}(\mathbf{W}\mathbf{D}^{-1})^T \\ &= \mathrm{tr}(\mathbf{D}^{-1}\mathbf{W}^T) = \mathrm{tr}(\mathbf{W}^T\mathbf{D}^{-1}) \\ &= -\mathrm{tr}(\mathbf{W}\mathbf{D}^{-1}) = -I_1. \end{aligned}$$

Hence $I_1 = 0$ and I_2 reduces to

$$I_2 = -\tfrac{1}{2}\,\mathrm{tr}(\mathbf{W}\mathbf{D}^{-1})^2. \tag{A6}$$

With $I_1 = I_3 = 0$, expansion of the determinant in (A4) leads to

$$(i\lambda)^3 + I_2(i\lambda) = 0. \tag{A7}$$

Hence the theorem is valid provided $I_2 > 0$.

To prove $I_2 > 0$, we diagonalize $\mathbf{D}^{-1}$ as

$$\mathbf{D}^{-1} = \mathbf{P}\Lambda\mathbf{P}^T, \qquad \mathbf{P}\mathbf{P}^T = \mathbf{I}, \tag{A8}$$

where Λ is a diagonal matrix with positive diagonal elements d_1, d_2, d_3. Then

$$\begin{aligned} I_2 &= -\tfrac{1}{2}\,\mathrm{tr}(\mathbf{W}\mathbf{P}\Lambda\mathbf{P}^T)^2 \\ &= -\tfrac{1}{2}\,\mathrm{tr}(\mathbf{W}\mathbf{P}\Lambda\mathbf{P}^T\mathbf{W}\mathbf{P}\Lambda\mathbf{P}^T) \\ &= -\tfrac{1}{2}\,\mathrm{tr}(\mathbf{P}^T\mathbf{W}\mathbf{P}\Lambda\mathbf{P}^T\mathbf{W}\mathbf{P}\Lambda) \\ &= -\tfrac{1}{2}\,\mathrm{tr}(\mathbf{W}^*\Lambda\mathbf{W}^*\Lambda), \end{aligned} \tag{A9}$$

in which $\mathbf{W}^* = \mathbf{P}^T\mathbf{W}\mathbf{P}$ is antisymmetric. Let

$$\mathbf{W}^* = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix}. \tag{A10}$$

Then it is readily shown that (A9) reduces to

$$I_2 = d_1d_2w_3^2 + d_2d_3w_1^2 + d_3d_1w_2^2 > 0. \tag{A11}$$

This completes the proof.